Hawks and Doves in a Dynamic Framework

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***Abstract*—We revisit in this paper the well-studied Hawk and Dove game within a dynamic framework. A non-standard evolutionary game approach is taken, in which the starting point of the modelling is the dynamic evolution of the populations as a function of the strategies used, instead of a fitness based model (in which the fitness functions determine the evolution). This work is motivated by the discussion in the book of Thomas L. Vincent co-authored with J. Brown [4] in which they raise (on page 73) the puzzling question of whom should one consider to be the players: the individuals or the populations?**

I. Introduction

Competition between Hawks and Doves has been used as one of the basic models for the evolution of some feature (aggressiveness in our case) within a population. The framework in which this model was studied the most is the one of evolutionary games, a branch of game theory developed by J. Maynard Smith [2]. This paradigm specializes in large populations in which the fraction of the different behaviours change as a result of large number of local pairwise interactions, i.e. interactions between randomly chosen pairs of individuals. Central in evolutionary games is the concept of Evolutionary Stable Strategy (ESS), which is a distribution of (deterministic or mixed) actions such that if used, the population is immune to penetration of mutations. This notion is stronger than that of Nash equilibrium as ESS is robust against a deviation of a whole fraction of the population where as the Nash equilibrium is defined with respect to possible deviations of a single player [4]. A second basic element in the theory of evolutionary games is that of the replicator dynamics, which provides the dynamic evolution of the ratio of the population that uses each action. There are various variants of the replicator dynamics, each giving distinct trajectories, that can be justified under appropriate conditions. We note that the ESS concept does not rely on how the dynamics are modelled, and therefore the modelling phase in evolutionary games often restricts to describing the local interactions between players along with the possible related fitness, and does not include global dynamic aspects of the populations.

In this paper, we propose a new modelling with several features that differs from the standard modelling of the Hawk and Dove game.

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- First, instead of defining a fitness (that each individual would optimize and then study the dynamics using some replicator dynamics), we start by including the dynamics of the population into the model and into the decision process.
- We supply each individual with an energy state and define how the "decision" of being aggressive or peaceful affect the transitions between the energy states. This decision is taken at birth genetically and it can't be changed (framework of an evolutionary game in [3]).
- Our approach differs from previous ones (where the global state is composed of the "fractions" of the population that adopt each type of behaviour) by proposing a model where the expected outcome of an interaction is determined not just by the fractions of the population in different states (against a randomly selected individual).
- We chose to focus on modelling of the whole population as players having a common objective. In this case, the rate of population growth depends on the number of individuals with a higher energy level. We do not assume that an individual inherits the behaviour of its parent.

Our goal here is to characterize the equilibrium policies as a function of the system parameters.

The structure of the paper is as follows. In Sec. II, we present the model and afterwards we focus on the population size dynamics. In Sec.III, we describe the evolutionary game and investigate the existence of ESS. We give explicit results (Sec. IV), on conditions for having pure or mixed equilibria. Finally, Sec. V gives a discussion on the framework of the paper. We conclude the paper in Sec. VI. Due to space limitations, all the proofs will be incorporated in the final version of the paper.

II. System Model

We consider a modified Hawk and Dove game with state dependent actions. The action taken by each individual depends on his local (individual) state. This state represents the energy like in [1]. We consider a three state model.

- In the highest state (denoted state 2 hereafter), an individual can act either as a *Hawk* (H) or a *Dove* (D) in order to get the resource. This action is decided at the birth of each individual. This means that each individual in state 2 acts either as H or D during all his life.

- In the intermediate state (denoted state 1), the only possible action is called W as an individual is weak. Every individual starts his life in this state.
- The last state corresponds to a dead individual.

Therefore, the life of an individual is governed by a three-state Markov process in which the transition rates depend on the action of the individual, genetically defined at his birth, and also the population profile and size.

The main difference with the model described in [1] is that the authors do not consider birth of individuals but penalties. Also, MacNamara et al. consider an infinite population of individuals but, in our model, we consider a finite number of individuals evolving depending on interactions.

A. Interactions

In the dynamic of the system, there are two possible events for an individual: meeting a resource with a rate α or losing energy (fall down to a lower individual state). An individual loses energy if he does not find any resource during a random amount of time. This time follows an exponential distribution with a parameter that depends on his action. If an individual plays Dove (resp. Hawk, Weak), this rate is γ_d (resp. γ_h, γ_1). If an individual gets a resource, then its local state goes from 1 to 2 (if the individual was in the intermediate state) or stay in state 2. An individual can lose energy by competing a resource with another individual and then his local state goes down from 2 to 1.

We have several scenarios of interactions between different individuals which leads to different transitions:

- If two Weak meet, then one of the two players gets the resource.
- If two Hawks meet, then one of the two gets the resource but the other loses energy.
- If a Dove/Weak meets a Hawk, the Hawk gets the resource and, the Dove/Weak is injured and falls down to the intermediate state.
- If a Dove meets a Weak, then the Dove gets the resource and there is no energy loss for both.
- If two Dove meet, then one of the two gets the resource and there is no energy loss for both.

B. Growth Rate Dynamics of Population Size

The originality of our approach consists in the size of the population which is assumed to evolve in time. We have two dynamics in our system: the population dynamics and the strategy dynamics.

Individuals are assumed to be born in the intermediate state with a rate λ. Only individuals in the highest state can give birth. Thus, the global birth rate is proportional to the number of individuals in the highest state. The individuals compete for resources by playing actions which depends on their local state. We assume that for each birth, there is a probability β that the new born individual does not follow the parents genes. It means that a Dove can give birth to a Hawk and conversely. Otherwise, a Dove (resp. a Hawk) gives birth to a Dove (resp. a Hawk). Each new undefined child chooses with probability u to be a Hawk for all his life.

We denote by $X_1^H(t)$ (resp. $X_2^H(t)$) the number of Hawk at time t in state 1 (resp. 2). We have the same notations $X_1^D(t)$ and $X_2^D(t)$ for the Doves. We denote by $X^H(t)$ (resp. $X^D(t)$) the total number of Hawks (resp. Doves) at time t. We note also by $X_1(t)$ (resp. $X_2(t)$) the total number of individuals in state 1 (resp. in state 2) at time t.

1) Hawk Population: We have the following differential equations for the dynamic of the Hawk population (individuals in state 1 and individuals in state 2).

$$\begin{aligned}\dot{X}_1^H(t) &= (1-\beta)\lambda_h X_2^H(t) + u\beta(\lambda_h X_2^H(t) + \lambda_d X_2^D(t)) \\ &\quad + \frac{\alpha(1-q(X))}{2} X_2^H(t) \frac{X_2^H(t)}{X} + \gamma_h X_2^H(t) \\ &\quad - \gamma_1 X_1^H(t) - \alpha q(X) X_1^H(t) \\ &\quad - \alpha(1-q(t)) \frac{1}{2} X_1^H(t) \frac{X_1^H(t)+X_1^D(t)}{X}. \\ \dot{X}_2^H(t) &= \alpha q(X) X_1^H(t) + \frac{\alpha(1-q(X))}{2} X_1^H(t) \frac{X_1^H(t)+X_1^D(t)}{X} \\ &\quad - \frac{\alpha(1-q(X))}{2} X_2^H(t) \frac{X_2^H(t)}{X} - \gamma_h X_2^H(t).\end{aligned} \tag{1}$$

We describe this process by:

- $(1-\beta)\lambda_h X_2^H(t)$: the total Hawk birth rate proportional to the number of Hawk in state 2 (with probability $(1-\beta)$ the new individual will be also a Hawk),
- $u\beta(\lambda_h X_2^H(t) + \lambda_d X_2^D(t))$: with a probability β, a new individual will not necessary follow his parents and, with probability u, he will act as an Hawk,
- $\frac{\alpha(1-q(X))}{2} X_2^H(t) \frac{X_2^H(t)}{X_1^H(t)+X_2^H(t)+X_1^D(t)+X_2^D(t)}$: when two Hawks fight for the resource, one of the two wins and the loser gets down to the intermediate state,
- $\gamma_h X_2^H(t)$: the rate that a Hawk in state 2 does not find resources,
- $\gamma_1 X_1^H(t)$: a Hawk in state 1 does not find a resource and dies.
- $\alpha q(X) X_1^H(t)$: when a Weak Hawk meets alone a resource, he gets it and gets up to the highest state,
- $\alpha(1-q(X)) \frac{1}{2} X_1^H(t) \frac{X_1^H(t)+X_1^D(t)}{X_1^H(t)+X_2^H(t)+X_1^D(t)+X_2^D(t)}$: when a Weak meets another Weak (Hawk or Dove) individual, one of the two gets the resource and gets up to the intermediate state.

We have the following relation:

$$\dot{X}_1^H(t) + \dot{X}_2^H(t) = 0,$$

which becomes:

$$(1-\beta)\lambda_h X_2^H(t) + u\beta(\lambda_h X_2^H(t) + \lambda_d X_2^D(t)) = \gamma_1 X_1^H(t).$$

2) Dove Population: We have the following differential equations for the dynamic of the Hawk population (individuals in state 1 and individuals in state 2).

$$
\begin{aligned}
\dot{X}_1^D(t) &= (1-\beta)\lambda_d X_2^D(t) \\
&+(1-u)\beta(\lambda_h X_2^H(t)+\lambda_d X_2^D(t)) \\
&+\alpha(1-q(X))X_2^D(t)\tfrac{X_2^H(t)}{X}+\gamma_d X_2^D(t) \\
&-\gamma_1 X_1^D(t)-\alpha q(X)X_1^D(t) \\
&-\alpha(1-q(X))\tfrac{1}{2}X_1^D(t)\tfrac{X_1^H(t)+X_1^D(t)}{X}. \\
\dot{X}_2^D(t) &= \alpha q(X)X_1^D(t) \\
&+\alpha(1-q(X))\tfrac{1}{2}X_1^D(t)\tfrac{X_1^H(t)+X_1^D(t)}{X} \\
&-\alpha(1-q(X))X_2^D(t)\tfrac{X_2^H(t)}{X}-\gamma_d X_2^D(t).
\end{aligned} \tag{2}
$$

We have the following relation:

$$\dot{X}_1^D(t)+\dot{X}_2^D(t)=0,$$

which becomes:

$$(1-\beta)\lambda_d X_2^D(t)+(1-u)\beta(\lambda_h X_2^H(t)+\lambda_d X_2^D(t))=\gamma_1 X_1^D(t).$$

We denote by $\tilde{X}(u)=\tilde{X}_1^H(u)+\tilde{X}_2^H(u)+\tilde{X}_1^D(u)+\tilde{X}_2^D(u)$ the stationary regime, i.e., the solution to the system composed of the four equations: $\dot{X}_1^H=0$; $\dot{X}_2^H=0$; $\dot{X}_1^D=0$ and $\dot{X}_2^D=0$. An analytical solution cannot be found in general, i.e., for all $u\in[0,1]$. However, for $u\in\{0,1\}$, this solution will be used to characterize the pure ESS states of the population in the following sections.

C. Probability of interactions

Assume that with a probability q an individual meets a resource alone. Then, the individual which is in the intermediate state, increases his energy and his local state becomes the highest one (state 2). This probability depends on the population size X. We assume that each individual has an opportunity to find an idle resource with a high probability. Hence, when the population size is $1+X$, this individual is in competition with X other individuals and the probability that he is alone is given by:

$$q(X)=\frac{1}{1+X},$$

where $X=X_1^H+X_1^D+X_2^H+X_2^D$ is the total size of the population.

D. Individual transition rates

Each individual is controlled by a three state Markov process. There is one process for each type of individual, i.e. one for Hawk and one for Dove. The transition rates between the states depend on the action of the individual which can be either Hawk or Dove; and also on the population profile u' (the proportion of Hawk in the population). We note that each individual is born in the state 1. Let $r_{i,j}(a,u')$ is the transition rate from state i to state j depending on the action $a\in\{D,H\}$.

If the individual is an Hawk, then the transition rates are:

- from 1 to 0, the rate is $r_{1,0}(H,u'):=\gamma_1$,
- from 1 to 2, the rate is $r_{1,2}(H,u'):=\alpha q(X)+\alpha(1-q(X))\frac{1}{2}\frac{\tilde{X}_1(u')}{\tilde{X}(u')}$,
- from 2 to 1, the rate is $r_{2,1}(H,u'):=\frac{\alpha(1-q(X))}{2}\frac{\tilde{X}_2^H(u')}{\tilde{X}(u')}+\gamma_h$.

If an individual is a Dove, then its transition rates are:

- from 1 to 0, the rate is $r_{1,0}(D,u'):=\gamma_1$,
- from 1 to 2, the rate is $r_{1,2}(D,u'):=\alpha q(X)+\alpha(1-q(X))\frac{1}{2}\frac{\tilde{X}_1(u')}{\tilde{X}(u')}$,
- from 2 to 1, the rate is $r_{2,1}(D,u') := \alpha(1-q(X))\frac{\tilde{X}_2^H(u')}{\tilde{X}(u')}+\gamma_d$.

The transition matrix of this Markov process is given by:

$$A(a,u')=\begin{pmatrix} 0 & 0 & 0 \\ \gamma_1 & -\gamma_1-r_{1,2}(a,u') & r_{1,2}(a,u') \\ 0 & r_{2,1}(a,u') & -r_{2,1}(a,u') \end{pmatrix}$$

Note that those transitions are equal even for a Dove and a Hawk, except the ones from the state 2 to the state 1. It means that $r_{1,2}(a,u')=r_{1,2}(u')$. This is obvious as the week individuals (individuals in state 1) behave similarly if it is a Hawk or a Dove.

III. Fitness and Formulation of the Game

A. Individual Fitness

The control parameter u determines the strategy of an individual. This parameter represents the probability for an individual to play Hawk in the highest state. We define by $J(a,u')$ the reward of an individual playing strategy $a\in\{H,D\}$ against the population profile u' (it means that u' is the proportion of Hawks in the population). Then we have

$$u'=\frac{\tilde{X}^H}{\tilde{X}}.$$

Each individual wants to maximize his expected time spent in state 2, as it is highly related to the average number of offspring of an individual. We denote by Z_t the state of a tagged individual at time t. The fitness of this individual taking action a is expressed by:

$$J(a,u') = \mathbb{E}_{a,u'}\int_0^\infty 1\!\!1_{\{Z_t=2\}}dt=\int_0^\infty p_t(a,u')dt, \tag{3}$$

where $p_t(a,u')=\mathbb{P}_{a,u'}(Z_t=2|Z_0=1)$. Since the expected number of births per individual is proportional to the time it spends in state 2, the fitness as we defined it, has the direct meaning of expected number of offspring of an individual.

Proposition 1: For all time t, the fitness of an individual playing action $a\in\{H,D\}$ against a population profile u' is given by:

$$J(a,u') = \frac{r_{1,2}(a,u')}{\gamma_1 r_{2,1}(a,u')}.$$

IV. The structure of the equilibria

In this section, we study the different possible structures of equilibria and under what conditions on the system's parameters they are obtained.

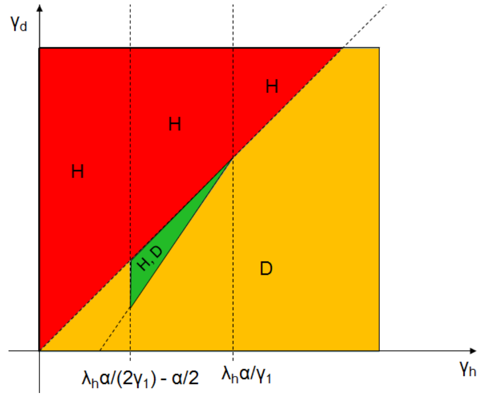

Fig. 1. Pure ESS as functions of γ_h and γ_d.

A. Pure equilibria

We first look at the existence of a pure equilibrium for our game. We have the two possible cases:

1) D is a pure equilibrium if and only if $J(0,0) > J(1,0)$.
2) H is a pure equilibrium if and only if $J(1,1) > J(0,1)$.

Proposition 2: *The strategy D is an ESS if and only if the expected remaining time in any state without getting resource of an individual playing "Dove", is strictly higher than the expected remaining time he gets playing "Hawk": $\frac{1}{\gamma_d} > \frac{1}{\gamma_h}$.*

Therefore, Dove is a pure ESS when the average amount of time an individual playing Dove can stay in state 2 without any new resource is higher than the same time characteristic to a Hawk individual. This result is somehow intuitive if we consider that an individual playing Hawk is more aggressive and may need more resources to survive.

Let us now consider the case where one individual is faced to a population of Hawk. In this case, we look for sufficient conditions on the model parameters that ensure a pure Hawk ESS.

Proposition 3: *The strategy H is an ESS if and only if one of the following condition is met:*

[C1] $\frac{\alpha\lambda_h - \gamma_1\gamma_h}{\alpha\gamma_1^2 + 2\gamma_1\gamma_h(\gamma_h+\gamma_1) - \alpha\lambda_h^2} < 0$ *and* $\gamma_d > \gamma_h$,

[C2] $\frac{\alpha\lambda_h - \gamma_1\gamma_h}{\alpha\gamma_1^2 + 2\gamma_1\gamma_h(\gamma_h+\gamma_1) - \alpha\lambda_h^2} \geq 0$ *and*
$\gamma_d > \gamma_h\left(1 + \frac{\gamma_1^2}{(\gamma_1+\lambda_h)^2}\right) - \frac{\alpha\gamma_1\lambda_h}{(\gamma_1+\lambda_h)^2}$.

The results in the previous propositions are illustrated in Fig. 1. In the following, we study the existence of a mixed ESS.

B. Mixed equilibria

In order to find a mixed ESS u^*, we have to solve the maximization problem:

$$u^* = \arg\max_{u\in[0,1]} J(u,u^*).$$

This optimization is equivalent to finding the probability u^* such that:

$$\int_0^\infty p_t(1,u^*)dt = \int_0^\infty p_t(0,u^*)dt.$$

Proposition 4: The mixed equilibrium u^* is solution of the following equation:

$$\frac{\widetilde{X}_2^H(u^*)}{1+\widetilde{X}(u^*)} = \frac{2(\gamma_h-\gamma_d)}{\alpha}.$$

This equilibrium exists if $\gamma_h - \frac{\alpha}{2} < \gamma_d < \gamma_h$.

V. Discussion

One can say that our model is not a real evolutionary game as all players have a common objective function. However, in such a game, a globally optimal solution is clearly a Nash equilibrium. It is in fact a *strong* equilibrium (the latter is defined as a multi-strategy for which no subset S of players can benefit if they all deviate). The converse is not true. For example, consider a game with two players, each having the compact interval $[0,1]$ as the strategy set, and a common utility given as the product of their actions. In this case, there are exactly two equilibria: $\{0,0\}$ and $\{1,1\}$. But only the latter one is globally optimal.

Now, consider a game where all players have common objectives. Here, a globally optimal solution need not be an ESS. In fact, an ESS may not exist and it might be advantageous to use the notion of ESS Set instead (see [5]). In our model, the fitness is the same for all individuals. However, the actions have an important influence on the birth rates and, therefore, on the dynamic of each sub-population.

VI. Concluding remarks

In this paper, we have revisited the Hawk and Dove evolutionary game. Our objective was to address the puzzling question raised in [4, p.73]. To this aim, we chose a framework which is between the two possibilities stated in the abstract: each individual is a player but the fitness of all players is the same (and does not depend on their type), i.e., all individuals have common goal. This allowed us to make use of the evolutionary game framework in which each individual is a player, rather than to consider the whole population as a single player.

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